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Price Uncertainty in Linear Production Situations

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Abstract

This paper analyzes linear production situations with price uncertainty, and shows that the corrresponding stochastic linear production games are totally balanced. It also shows that investment funds, where investors pool their individual capital for joint investments in financial assets, fit into this framework. For this subclass, the paper provides a procedure to construct an optimal investment portfolio. Furthermore it provides necessary and sufficient conditions for the proportional rule to result in a core-allocation.

KEYWORDS: linear production, stochastic cooperative games, investment funds.
JEL-codes: C71.

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1 Introduction

Production processes are classic examples of situations where several parties recognize the benefits of cooperation. The owners of the production factors labor, resources, capital, and technology join forces to deliver a product or a service that is valued more by consumers than their separate inputs. The value that is added by production is the benefit of cooperation and has to be divided between the parties that are involved in the production process.

Owen (1975) was the first to analyze this problem from a game theoretical point of view. In his model, agents have their own individual bundle of resources which they can use as inputs for a publicly available linear production technology. The resulting output can be sold on the market for given prices. In this situation, agents can individually use their resources to maximize the proceeds from production, but they can probably do better if they cooperate with each other and combine their resources. For instance, an agent that lacks certain inputs for production, prefers cooperating with agents that possess the required inputs, so that production can take place. Owen (1975) shows that the corresponding linear production games are totally balanced. In particular, he shows that a core-allocation arises if each agent receives the marginal value of his resources. Here, the marginal value of a resource is the marginal revenue generated by one extra unit of this resource.

The analysis of Owen (1975) as well as most subsequent studies on this subject, confine themselves to a deterministic setting. Real production, however, typically features uncertainty. Due to irregularities in the production process, the quality of produced output may not always be up to standard so that production losses may occur. If this is the case, there is uncertainty about the output. Similarly, when production takes a considerable amount of time, there is uncertainty about the price at the moment that the production decision is made.

Sandsmark (1999) examines the cooperative behavior in a two stage production model with uncertainty about the output. The first stage production plan is determined under uncertainty while the second stage production plan may be contingent on the realized state of nature. It is shown that the resulting cooperative game, in which coalitions maximize a certain revenuefunction, has a nonempty core. Furthermore, a specific core-allocation is provided. Similar to chance-constrained games (cf. Charnes and Granot (1973)), this model does not explicitly take into account the individual preferences of the agents. The revenuefunction, however, may be interpreted as the sum of individual expected utilities.

In contrast to Sandsmark (1999), this paper analyzes linear production situations with price uncertainty. When agents decide upon their production plan, prices are still unknown. As a result, price volatility may play a role in deciding how agents use their resources. For instance, they may use their resources to produce different outputs so as to reduce the variability in total revenues. We model linear production situations with price uncertainty by means of stochastic cooperative games and show that the corresponding stochastic linear production games are balanced. For an extensive discussion of stochastic cooperative games we refer to Suijs (1999).

Linear production games with price uncertainty can be used to describe investment problems, where agents can apply their (individual) capital to invest in financial assets whose future value is uncertain at the time of investment. We discuss this application more detailed in a separate section.

2 Stochastic Cooperative games

Let us first recall some of the definitions concerning stochastic cooperative games as introduced by Suijs, Borm, De Waegenaere and Tijs (1999). A stochastic cooperative game is described by a tuple $\Gamma = (N, \{\mathcal{X}_S\}_{S\subset N}, \{\succsim_i\}_{i\in N})$, where N is the set of agents, $\mathcal{X}_S$ the nonempty set of random payoffs coalition S can obtain, and $\succsim_i$ the preference relation of agent i over the set $L^1(\mathbb{R})$ of stochastic payoffs with finite expectation. We assume that for each agent the preferences are complete, transitive and continuous[1]. The class of all cooperative games with stochastic payoffs with agent set N is denoted by $SG(N)$. For a more extensive discussion of this model and some examples we refer to Suijs et al. (1999) and Suijs, De Waegenaere and Borm (1998).

An allocation of a stochastic payoff $X_S \in \mathcal{X}_S$ to coalition S is described by a pair $(d, r) \in \mathbb{R}^S \times \mathbb{R}^S$ such that $\sum_{i\in S} d_i \leq 0$ and $\sum_{i\in S} r_i = 1$ and $r_i \geq 0$ for all $i \in S$. The payoff to agent $i \in S$ according to the allocation (d, r) equals $d_i + r_i X_S$. The set of all allocations for coalition S is denoted by $Z_\Gamma(S)$.

The core of a stochastic cooperative game is defined as follows. Let $\Gamma \in SG(N)$ and $(d_i + r_i X_N)_{i\in N} \in Z_\Gamma(N)$. Then the allocation $(d_i + r_i X_N)_{i\in N}$ is a core allocation for the game Γ if for each coalition S there is no allocation $(\tilde{d}_i + \tilde{r}_i X_N)_{i\in S} \in Z_\Gamma(S)$ such that $\tilde{d}_i + \tilde{r}_i X_S \succ_i d_i + r_i X_S$ for all $i \in S$. The set of all core allocations for Γ is denoted by $C(\Gamma)$.

Next, consider preferences $\{\succsim_i\}_{i\in N}$ such that for each $i \in N$ there exists a function $m_i : L^1(\mathbb{R}) \to \mathbb{R}$ satisfying

(M1) for all $X, Y \in L^1(\mathbb{R}) : X \succsim_i Y$ if and only if $m_i(X) \geq m_i(Y)$;

(M2) for all $X \in L^1(\mathbb{R})$ and all $d \in \mathbb{R}$: $m_i(d + X) = d + m_i(X)$.

The interpretation is that $m_i(X)$ equals the amount of money m for which agent i is indifferent between receiving the amount $m_i(X)$ with certainty and receiving the stochastic payoff X. The amount $m_i(X)$ is called the certainty equivalent of X. Condition (M1) states that agent i weakly prefers one stochastic payoff to another one if and only if the certainty equivalent of the former is greater than or equal to the certainty equivalent of the latter. Condition (M2) states that the certainty equivalent is linearly separable in the deterministic amount of money

[1] The preferences $\succsim$ are continuous if for all $X \in L^1(\mathbb{R})$ the sets $\{Y \in L^1(\mathbb{R}) | Y \succsim X\}$ and $\{Y \in L^1(\mathbb{R}) | Y \precsim X\}$ are closed.

d. The class of all stochastic coopertive games satisfying conditions (M1) and (M2) is denoted by $MG(N)$.

Example 1 Consider the preferences based on a utility function of the form $U(t) = \beta e^{-\alpha t}$, $(t \in \mathbb{R})$, where $\beta < 0$ and $\alpha > 0$. The certainty equivalent of $X \in L^1(\mathbb{R})$ can be defined by $m(X) = U^{-1}(E(U(X)))$. It is easy to check that m satisfies condition (M1). For condition (M2), let $X \in L^1(\mathbb{R})$ and $d \in \mathbb{R}$. Then $U^{-1}(t) = -\frac{1}{\alpha}\log\left(\frac{t}{\beta}\right),\ t < 0$ and

$$\begin{aligned} m(d+X) &= U^{-1}(E(U(d+X))) \\ &= -\tfrac{1}{\alpha}\log\left(\tfrac{1}{\beta}\int \beta e^{-\alpha(d+t)}dF_X(t)\right) \\ &= -\tfrac{1}{\alpha}\log\left(e^{-\alpha d}\tfrac{1}{\beta}\int \beta e^{-\alpha t}dF_X(t)\right) \\ &= d - \tfrac{1}{\alpha}\log\left(\tfrac{1}{\beta}(\int \beta e^{-\alpha t}dF_X(t))\right) \\ &= d + m(X). \end{aligned}$$

The following theorem regarding nonemptiness of the core is due to Suijs and Borm (1999).

Theorem 2 Let $\Gamma \in MG(N)$. Then $C(G) \neq \emptyset$ if and only if $C(v_\Gamma) \neq \emptyset$, where

$$v_\Gamma(S) = \max\left\{\sum_{i\in S} m_i(\hat{d}_i + \hat{r}_i\hat{X}) |\ (\hat{d}_i + \hat{r}_i\hat{X})_{i\in S} \in Z_\Gamma(S)\right\},$$

for all $S \subset N$.

3 Stochastic Linear Production Games

Let $N \subset \mathbb{N}$ denote the set of agents, $R \subset \mathbb{N}$ the set of resources, and $M \subset \mathbb{N}$ the set of consumption goods. By assumption, each agent $i \in N$ is a risk averse expected utility maximizer with utility function $U_i(t) = \beta_i e^{-\alpha_i t}$, where $\beta_i < 0$ and $\alpha_i > 0$. Let $b^i \in \mathbb{R}_+^R$ denote agent i's endowment of resources, and let $A^i \in \mathbb{R}^{R\times M}$ denote the linear production technology of agent i. This means that for the production of an output bundle $c \in \mathbb{R}_+^R$ the inputs $A^i c \in \mathbb{R}_+^M$ are needed. Note that this generalizes Owen (1975), since in that case each agent has access to the same technology $A \in \mathbb{R}^{R\times M}$. The resources can be used to produce consumption goods. The prices at which these goods can be sold, are denoted by random variables $P_j \in L^1(\mathbb{R}_+)$, $j \in M$, and are assumed to be mutually independent.

When coalition S forms, the members of S pool their resources and production technologies, so that they possess the resources $\sum_{i\in S} b^i$ and have access to the technologies A^i,

$i \in S$. Let $c^i \in \mathbb{R}_+^R$ denote the consumption bundle that coalition S plans to produce with the production technology A^i of agent $i \in S$. Then a production plan $c \in \mathbb{R}_+^R$ is feasible for coalition S if it possesses the necessary resources, that is there exist separate production plans $(c^i)_{i\in S}$ such that $\sum_{i\in S} c^i = c$ and $\sum_{i\in S} A^i c^i \leq \sum_{i\in S} b^i$. The set of feasible production plans for coalition $S \subset N$ thus equals

$$C(S) \;=\; \left\{ \sum_{i\in S} c^i \,\middle|\, \forall_{i\in S} : c^i \in \mathbb{R}_+^R,\, \sum_{i\in S} A^i c^i \leq \sum_{i\in S} b^i \right\}. \tag{1}$$

Each feasible production plan $c \in C(S)$ yields stochastic revenues $\sum_{j\in M} P_j c_j$. Now, we can describe a linear production situation with price uncertainty by the following stochastic linear production game $\Gamma = (N, \{\mathcal{X}_S\}_{S\subset N}, \{\succsim\}_{i\in N})$, with

$$\mathcal{X}_S \;=\; \left\{ \sum_{j\in M} P_j c_j \,\middle|\, c \in C(S) \right\} \tag{2}$$

for all $S \subset N$ and $\succsim_i$ the preferences induced by U_i, for each $i \in N$. The set of all stochastic linear production games is denoted by $SLP(N)$.

Example 1 implies that stochastic linear production games belong to the class $MG(N)$. Hence, for our analysis we may focus on certainty equivalents. Therefore, let $\Gamma \in SLP(N)$ be a stochastic linear production game. Then, given a feasible production plan $c \in C(S)$, the certainty equivalent of an allocation $\left(d_i + r_i \sum_{j\in M} P_j c_j\right)_{i\in S}$ is for each agent $i \in S$ equal to

$$\begin{aligned} m_i\left(d_i + r_i \sum_{j\in M} P_j c_j\right) \;&=\; U_i^{-1}\left(E(U_i(d_i + r_i \sum_{j\in M} P_j c_j))\right) \\ &=\; d_i + \sum_{j\in M} U_i^{-1}\left(E\left(U_i\left(r_i P_j c_j\right)\right)\right) \\ &=\; d_i + \sum_{j\in M} \tfrac{1}{\alpha_i} \sum_{j\in M} \log\left(\int_0^\infty e^{-\alpha_i r_i c_j t} dF_{P_j}(t)\right)^{-1}, \end{aligned}$$

where the second equality follows from the independence of $(P_j)_{j\in M}$. The corresponding TU-game (N, v_Γ) is then given by

$$\begin{aligned} v_\Gamma(S) \;=\; \max \; & \sum_{i\in S} \tfrac{1}{\alpha_i} \sum_{j\in M} \log\left(\int_0^\infty e^{-\alpha_i r_i c_j t} dF_{P_j}(t)\right)^{-1} \\ \text{s.t.:} \quad & \textstyle\sum_{i\in S} r_i \;=\; 1, \\ & r_i \;\geq\; 0, \quad \text{for all } i \in S, \\ & c \in C(S), \end{aligned} \tag{3}$$

for all $S \subset N$.

We cannot explicitly determine an optimal production plan $c \in C(S)$. What we can do though, is determine the Pareto optimal allocation r of the random revenue $\sum_{j\in M} P_j c_j$, while taking the production plan $c \in C(S)$ as given.

Proposition 1 Let $\Gamma \in SLP(N)$ and let $c \in C(S)$ be a feasible production plan for coalition $S \subset N$. An allocation $\left(d_i + r_i \sum_{j\in M} P_j c_j\right)_{i\in S}$ is a Pareto optimal allocation of the revenue $\sum_{j\in M} P_j c_j$ if and only if

$$r_i^* = \frac{\frac{1}{\alpha_i}}{\sum_{j\in S}\frac{1}{\alpha_j}} \tag{4}$$

for all $i \in S$.

PROOF: Suijs and Borm (1999) show that an allocation $(d_i^* + r_i^* \sum_{j\in M} P_j c_j)_{i\in S}$ is Pareto optimal, if and only if r^* is the solution to

$$\begin{aligned} &\max \sum_{i\in S} \tfrac{1}{\alpha_i} \sum_{j\in M} \log\left(\int_0^\infty e^{-\alpha_i r_i c_j t} dF_{P_j}(t)\right)^{-1} \\ &\text{s.t.:} \quad \textstyle\sum_{i\in S} r_i = 1, \\ &\qquad\quad r_i \geq 0, \quad \text{for all} i \in S. \end{aligned}$$

From Lemma 1 with $c = -\alpha_i c_j$ and $x = r_i$ it follows that the objective function is strictly concave, so that the optimal solution is unique. That r^* is indeed the unique solution follows from the fact that it satisfies the Karush-Kuhn-Tucker conditions:[2]

$$\begin{aligned} -\sum_{j\in M} c_j \frac{\int_0^\infty t e^{-\alpha_i r_i^* c_j t} dF_{P_j}(t)}{\int_0^\infty e^{-\alpha_i r_i^* c_j t} dF_{P_j}(t)} &= \lambda - \nu_i, \\ \nu_i r_i &= 0, \qquad \text{for all } i \in S, \end{aligned}$$

with $\nu_i = 0$ for all $i \in S$,

$$\lambda = -\sum_{j\in M} c_j \frac{\int_0^\infty t e^{-c_j t/\alpha(S)} dF_{P_j}(t)}{\int_0^\infty e^{-c_j t/\alpha(S)} dF_{P_j}(t)},$$

and $\alpha(S) = \sum_{i\in S} \frac{1}{\alpha_i}$. □

Note that more risk averse agents bear a larger part of the risk. Furthermore, note that the Pareto optimal allocation r^* is independent of the production plan $c \in C(S)$, so that r^* also yields a Pareto optimal allocation for the optimal production plan.

[2] The Karush-Kuhn-Tucker conditions read as follows:

$$\begin{aligned} \text{If } f(x) &= \max_y f(y) \\ &\quad \text{s.t. } g_k(y) \leq 0, \quad k \in K \\ &\qquad\;\; g_l(y) = 0, \quad l \in L \end{aligned}$$

then there exist $\nu_k \geq 0$ $(\forall k \in K)$ and $\lambda_l \in \mathbb{R}$ $(\forall l \in L)$ such that

$$\begin{aligned} \nabla f(x) &= \textstyle\sum_{k\in K} \nu_k \cdot \nabla g_k(x) + \sum_{l\in L} \lambda_l \cdot \nabla g_l(x) \\ \nu_k \cdot g_k(x) &= 0, \text{ for all } k \in K. \end{aligned}$$

Moreover, if f is strictly concave and g_k $(k \in K)$, g_l $(l \in L)$ are convex then the reverse of the statement also holds and the maximum is unique.

The following theorem shows that stochastic linear production games are totally balanced.

Theorem 2 Each stochastic linear production game $\Gamma \in SLP(N)$ is totally balanced.

PROOF: Let $\Gamma \in SLP(N)$. Since each subgame $\Gamma_{|S} \in SLP(S)$, it suffices to show that Γ is balanced. From Theorem 2 we know that it suffices to show that the TU-game (N, v_Γ) has a nonempty core. For this, we apply the well-known result of Bondareva (1963) and Shapley (1967). First, we substitute (4) in expression (3) and derive that

$$v_\Gamma(S) = \max \left\{ \alpha(S) \sum_{j\in M} \log\left(\int_0^\infty e^{-c_j t/\alpha(S)} dF_{P_j}(t) \right)^{-1} \middle| \, c \in C(S) \right\}. \tag{5}$$

Second, let λ be a balanced map[3] and denote the optimal production plan for coalition S by $c^S \in C(S)$. In particular, let $(c^{S,i})_{i\in S}$ be such that $\sum_{i\in S} c^{S,i} = c^S$ and $\sum_{i\in S} A^i c^{S,i} \leq \sum_{i\in S} b^i$. So, $c^{S,i}$ is that part of the production plan that is produced with the technology A^i. Since

$$\begin{aligned}
\sum_{i\in S} A^i \sum_{S\subset N} \lambda(S) c^{S,i} &= \sum_{S\subset N} \lambda(S) \sum_{i\in S} A^i c^{S,i} \\
&\leq \sum_{S\subset N} \lambda(S) \sum_{i\in S} b^i \\
&= \sum_{i\in N} \sum_{S\subset N: i\in S} \lambda(S) b^i \\
&= \sum_{i\in N} b^i,
\end{aligned}$$

it holds that $\sum_{S\subset N} \lambda(S) c^S \in C(N)$. Next, we derive that

$$\begin{aligned}
\sum_{S\subset N} \tfrac{\lambda(S)}{\alpha(N)} v_\Gamma(S) &= \sum_{S\subset N} \tfrac{\lambda(S)\alpha(S)}{\alpha(N)} \sum_{j\in M} \log\left(\int_0^\infty e^{-c_j^S t/\alpha(S)} dF_{P_j} \right)^{-1} \\
&\leq \sum_{j\in M} \log\left(\int_0^\infty e^{-\sum_{S\subset N} \lambda(S)\alpha(S)/\alpha(N)\cdot c_j^S t/\alpha(S)} dF_{P_j}(t) \right)^{-1} \\
&= \sum_{j\in M} \log\left(\int_0^\infty e^{-\sum_{S\subset N} \lambda(S) c_j^S t/\alpha(N)} dF_{P_j}(t) \right)^{-1},
\end{aligned} \tag{6}$$

where the inequality follows from $\sum_{S\subset N} \frac{\lambda(S)\alpha(S)}{\alpha(N)} = 1$ and Lemma 1, which states that the objective function is concave. Balancedness then follows from

$$\begin{aligned}
\sum_{S\subset N} \lambda(S) v_\Gamma(S) &\leq \alpha(N) \sum_{j\in M}^{m} \log\left(\int_0^\infty e^{-\sum_{S\subset N} \lambda(S) c_j^S t/\alpha(N) dF_{P_j}(t)} \right)^{-1} \\
&\leq \alpha(N) \sum_{j\in M} \log\left(\int_0^\infty e^{-c_j^N t/\alpha(N)} dF_{P_j}(t) \right)^{-1} \\
&= v_\Gamma(N),
\end{aligned}$$

[3] A balanced map l is a function $\lambda : \mathbb{R}^{2^N} \to \mathbb{R}_+$ such that $\sum_{i\in N} \lambda(S) = 1$ for all $i \in N$.

where the first and second inequality follow from (6) and $\sum_{S \subset N} \lambda(S) c^S \in C(N)$, respectively.
$\square$

With deterministic prices, a core-allocation can be found by appropriately valuating the different resources, and giving each agent the total value of his own resource bundle. To see how this process works, recall that a deterministic linear production game (N, v) (cf. Owen (1975)) is given by

$$v(S) = \max \left\{ p^\top c \,\middle|\, Ac \leq \sum_{i \in S} b^i \right\} \tag{7}$$

for all $S \subset N$. The dual optimization problem of (7) for coalition N is given by

$$\min \left\{ x^\top \sum_{i \in N} b^i \,\middle|\, x^\top A \geq p \right\}.$$

Let $\xi \in \mathbb{R}_+^R$ be the optimal solution of the dual problem. Then ξ_k represents the marginal value of resource k, that is one extra unit of resource k raises the value $v(N)$ with ξ_k. One can interpret ξ_k as the monetary value of owning one unit of resource k, and $\sum_{k \in R} \xi_k b_k$ as the total value of owning the resource bundle $b \in \mathbb{R}_+^R$. Owen (1975) shows that a core-allocation arises for the linear production game defined in (7), if each agent $i \in N$ receives the total monetary value $\sum_{k \in R} \xi_k b_k^i$ of his individual resource bundle b^i.

This naturally raises the question if we can construct core-allocations in a similar way for stochastic linear production games? The answer is no. In general, there exists no $\xi \in \mathbb{R}_+^R$ such that $\left(\xi^\top b^i\right)_{i \in N}$ is a core-allocation for the stochastic linear production game (N, v_Γ) as defined in (3) and (5).

Example 3 Consider the following two person stochastic linear production game $\Gamma \in SLP(N)$, with $\alpha_1 = 0.1$, $\alpha_2 = 10$, $A = [1]$, $b^1 = 0.1$, $b^2 = 1$, and P_1 exponentially distributed with parameter 0.5. Note that the production technology uses one unit of input to produce one unit of output. The corresponding TU-game (N, v_Γ) equals $v_\Gamma(\{1\}) = 0.198$, $v_\Gamma(\{2\}) = 1.019$, and $v_\Gamma(\{1,2\}) = 1.990$. Now, let $\xi \geq 0$ denote the monetary value of the single resource. If $(\xi b^1, \xi b^2)$ belongs to the core of (N, v_Γ), it holds that $\xi b^1 = 0.1\xi \geq 0.198 = v_\Gamma(\{1\})$, $\xi b^2 = \xi \geq 1.019 = v_\Gamma(\{2\})$, and $\xi b^1 + \xi b^2 = 1.1\xi = 1.990 = v_\Gamma(\{1,2\})$. Since the first inequality implies that $\xi \geq 1.98$, we obtain the contradiction that $1.1\xi \geq 2.178 > 1.990 = v_\Gamma(\{1,2\})$. Hence, we cannot obtain a core-allocation by appropriately valuating the resource.

Theorem 4 Let $\Gamma \in SLP(N)$. If the resource bundles $(b^i)_{i \in N}$ are linearly independent, then there exists a vector $\xi \in \mathbb{R}^R$ such that $(\xi^\top b^i)_{i \in N} \in C(v_\Gamma)$.

PROOF: Take $\Gamma \in SLP(N)$ and let $\xi \in \mathbb{R}^R$. Then $(\xi^\top b^i)_{i \in N} \in C(v_\Gamma)$ if

$$
\begin{aligned}
&-\xi^\top \sum_{i\in S} b^i \leq -v_\Gamma(S) \quad \text{for all } S \subset N,\\
&\xi^\top \sum_{i\in N} b^i \geq v_\Gamma(N).
\end{aligned} \tag{8}
$$

By using a variant of Farka's Lemma[4], such ξ exists if and only if there exist no $\mu \geq 0$, $\lambda(S) \geq 0$ for all $S \subset N$, such that

$$
\begin{aligned}
&\sum_{S\subset N} -\lambda(S) \sum_{i\in S} b^i + \mu \sum_{i\in N} b^i = 0,\\
&\sum_{S\subset N} -\lambda(S) v_\Gamma(S) + \mu v_\Gamma(N) < 0.
\end{aligned}
$$

Rearranging terms yields that there exist no $\mu \geq 0$, $\lambda(S) \geq 0$ for all $S \subset N$, satisfying

$$
\begin{aligned}
&\sum_{i\in N} b^i \sum_{S\subset N: i\in S} \lambda(S) = \mu \sum_{i\in N} b^i,\\
&\sum_{S\subset N} \lambda(S) v_\Gamma(S) > \mu v_\Gamma(N).
\end{aligned}
$$

Since $\lambda(S) \geq 0$, $v_\Gamma(S) \geq 0$ for all $S \subset N$, and $b^i \geq 0$ for all $i \in N$, it follows that $\mu > 0$. Hence, without loss of generality we may assume that $\mu = 1$, that is there exist no $\lambda(S) \geq 0$ for all $S \subset N$ such that

$$
\begin{aligned}
&\sum_{i\in N} b^i \sum_{S\subset N: i\in S} \lambda(S) = \sum_{i\in N} b^i,\\
&\sum_{S\subset N} \lambda(S) v_\Gamma(S) > v_\Gamma(N).
\end{aligned}
$$

From the linear independence of $(b^i)_{i\in N}$ it follows that $\sum_{i\in N} b^i z_i = \sum_{i\in N} b^i$ has a unique solution $z_i = 1$ for all $i \in N$. This implies that $\sum_{S\subset N: i\in S} \lambda(S) = z_i = 1$ for all $i \in N$, and hence, that λ is a balanced map. The balancedness of (N, v_Γ) then yields the contradiction $\sum_{S\subset N} \lambda(S) v_\Gamma(S) \leq v_\Gamma(N)$. Conclusion, there exists a vector $\xi \in \mathbb{R}^R$ satisfying (8). □

Although linear independence is a sufficient condition to guarantee the existence of an appropriate valuation scheme of the resources, this result cannot be obtained without allowing for negative monetary values, as the following example shows.

Example 5 Consider the stochastic linear production game presented in Example 3, but now with

$$
A = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ b^1 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix},\ \text{and } b^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
$$

Note that the second resource is not needed for production, so that the corresponding TU-game (N, v_Γ) is the same as in Example 3. Since the resource bundles are linearly independent, we can find values $\xi = (\xi_1, \xi_2)$ such that $(\xi^\top b^1, \xi^\top b^2) \in C(v_\Gamma)$. Recall that $\xi_1 \geq 1.98$ and that $\xi^\top b^1 + \xi^\top b^2 = 1.1\xi_1 + \xi_2 = v_\Gamma(\{1,2\}) = 1.99$. Since $\xi_2 = 1.99 - 1.1\xi_1 \leq -0.188$, the superfluous resource must have a negative monetary value.

[4] The system $Ax \leq b$ has a solution x if and only if there exists no $y \geq 0$ such that $y^\top A = 0$ and $y^\top b < 0$.

4 Financial Games

The fact that retail prices are uncertain when individuals decide upon their production plan, is particularly applicable to investment funds. For at the time that people invest their capital in financial assets, they are not sure about the future value of this asset. To reduce the volatility of the total returns, they may prefer to participate in an investment fund to obtain a more diversified portfolio.

Besides reducing volatility, people may also benefit from cooperation if the return varies with the amount of capital invested. Bank deposits, for instance, usually earn a higher interest rate when more capital is deposited. These problems were introduced by Lemaire (1983) and further analyzed by Izquierdo and Rafels (1996) and Borm, De Waegenaere, Rafels, Suijs, Tijs and Timmer (1999). The latter extended the former model by considering several periods so as to include term- dependent interest rates. In contrast to the model we present here, the three previous models abstract from risk bearing investments. Instead, they assume that the earned interest rates are known with certainty beforehand.

In the financial games we consider, there is a society $N \subset \mathbb{N}$ of risk averse expected utility maximizing agents, each having a utility function of the form $U_i(t) = \beta_i e^{-\alpha_i t}$ with $\beta_i < 0$ and $\alpha_i > 0$. Each agent has capital $\omega^i \in \mathbb{R}_+$ available for investments in several, infinitely divisible risky assets $M \subset \mathbb{N}$. For each asset $j \in M$, let $\pi_j \in \mathbb{R}_+$ denote the asset price, $R_j \in L^1(\mathbb{R}_+)$ the random future value, and $q_j \in \mathbb{R}_+$ the quantity purchased. Then a portfolio $q \in \mathbb{R}_+^M$ of assets is feasible for coalition $S \subset N$ if they have sufficient capital at their disposal, that is if $\sum_{j \in M} \pi_j q_j \leq \sum_{i \in S} \omega_i$. Furthermore, the revenue of a portfolio $q \in \mathbb{R}_+^R$ equals $\sum_{j \in M} R_j q_j$. Since agents are not forced to invest all their capital in risky assets, we must assume that there is a riskless asset $j_0 \in M$ that earns the risk-free interest rate $r \geq 0$. Furthermore, we may assume without loss of generality that $\pi_{j_0} = 1$.

Summarizing, a financial game $\Gamma = (N, \{\mathcal{X}_S\}_{S \subset N}, \{\succsim_i\}_{i \in N})$ is described by

$$\mathcal{X}_S = \left\{ \sum_{j \in M} R_j q_j \,\middle|\, \exists_{q \in \mathbb{R}_+^M} : \sum_{j \in M} \pi_j q_j \leq \sum_{i \in S} \omega_i \right\}, \tag{9}$$

for all $S \subset N$, where the preferences $\succsim_i$ are induced by U_i. The class of financial games is denoted by $FG(N)$.

It is a straightforward exercise to see that $FG(N) \subset SLP(N)$. For investing in a risky asset can be described by a simple production technology, that uses a single resource, i.e. capital, to produce several commodities, i.e. financial assets. The production technology is determined by the asset prices $(\pi_j)_{j \in M}$: the asset price π_j denotes how much of the resource capital is needed to produce one unit of asset $j \in M$. The prices at which produced output can be sold are the random returns R_j. Hence, we obtain the following result.

Theorem 1 Each financial game $\Gamma \in FG(N)$ is totally balanced.

4.1 The Optimal Investment Portfolio

The relatively simple production technology in financial games enables us to provide a more detailed analysis of the optimal investment portfolio. Using the Pareto optimality result of Proposition 1, the optimal portfolio q^* of coalition S is the optimal solution of

$$\begin{array}{lrcll} \max & \multicolumn{4}{l}{\alpha(S) \sum_{j \in M} \log \left(\int_0^\infty e^{-q_j t/\alpha(S)} dF_{R_j}(t) \right)^{-1}} \\ \text{s.t.:} & \sum_{j \in M} \pi_j q_j & \leq & \sum_{i \in S} \omega^i, & \\ & q_j & \geq & 0, & \text{for all } j \in M. \end{array} \tag{10}$$

Recall that there is a risk-free asset j_0 yielding the risk-free interest rate $r \geq 0$. Hence, (10) is equivalent to

$$\begin{array}{lrcll} \max & \multicolumn{4}{l}{(1+r) q_{j_0} \; + \; \alpha(S) \sum_{j \in M_0} \log \left(\int_0^\infty e^{-q_j t/\alpha(S)} dF_{R_j}(t) \right)^{-1}} \\ \text{s.t.:} & \sum_{j \in M} \pi_j q_j & \leq & \sum_{i \in S} \omega^i, & \\ & q_j & \geq & 0, & \text{for all } j \in M, \end{array}$$

where $M_0 = M \backslash \{j_0\}$. Instead of focusing on the quantities, we can also focus on the capital invested in each asset. For this purpose, let $w_j = \pi_j q_j$ denote the capital invested in asset $j \in M$. Then substituting $q_j = w_j/\pi_j$ yields

$$\begin{array}{lrcll} \max & \multicolumn{4}{l}{(1+r) w_{j_0} \; - \; \alpha(S) \sum_{j \in M_0} \log \left(\int_0^\infty e^{-w_j/\pi_j \cdot t/\alpha(S)} dF_{R_j}(t) \right)} \\ \text{s.t.:} & \sum_{j \in M} w_j & \leq & \sum_{i \in S} \omega^i, & \\ & w_j & \geq & 0, & \text{for all } j \in M. \end{array}$$

$$= \begin{array}{lrcll} \max & \multicolumn{4}{l}{(1+r) w_{j_0} \; + \; \sum_{j \in M_0} \int_0^{w_j} \frac{1}{\pi_j} \frac{\int_0^\infty t e^{-x_j/\pi_j \cdot t/\alpha(S)} dF_{R_j}(t)}{\int_0^\infty e^{-x_j/\pi_j \cdot t/\alpha(S)} dF_{R_j}(t)} dx_j} \\ \text{s.t.:} & \sum_{j \in M} w_j & \leq & \sum_{i \in S} \omega_i, & \\ & w_j & \geq & 0, & \text{for all } j \in M. \end{array}$$

Since coalition S prefers investing in the risk-free asset to not investing at all, all the capital is spent in the optimal portfolio. Thus, we may replace $\sum_{j \in M} w_j^* \leq \sum_{i \in S} \omega^i$ by $\sum_{j \in M} w_j = \sum_{i \in S} \omega^i$. Substituting $w_{j_0} = \sum_{i \in S} \omega^i - \sum_{j \in M_0} w_j$ gives

$$\begin{array}{lrcll} \max & \multicolumn{4}{l}{(1+r) \left(\sum_{i \in S} \omega^i - \sum_{j \in M_0} w_j \right) \; + \; \sum_{j \in M_0} \int_0^{w_j} \frac{1}{\pi_j} \frac{\int_0^\infty t e^{-x_j/\pi_j \cdot t/\alpha(S)} dF_{R_j}(t)}{\int_0^\infty e^{-x_j/\pi_j \cdot t/\alpha(S)} dF_{R_j}(t)} dx_j} \\ \text{s.t.:} & \sum_{j \in M_0} w_j & \leq & \sum_{i \in S} \omega^i, & \\ & w_j & \geq & 0, & \text{for all } j \in M_0. \end{array}$$

$$= \begin{array}{lrcll} \max & \multicolumn{4}{l}{(1+r) \sum_{i \in S} \omega^i \; + \; \sum_{j \in M_0} \int_0^{w_j} \frac{1}{\pi_j} \frac{\int_0^\infty t e^{-x_j/\pi_j \cdot t/\alpha(S)} dF_{R_j}(t)}{\int_0^\infty e^{-x_j/\pi_j \cdot t/\alpha(S)} dF_{R_j}(t)} - (1+r) dx_j} \\ \text{s.t.:} & \sum_{j \in M_0} w_j & \leq & \sum_{i \in S} \omega^i, & \\ & w_j & \geq & 0, & \text{for all } j \in M_0. \end{array}$$

$$= \quad \max\, (1+r)\sum_{i\in S}\omega^i \; + \sum_{j\in M_0}\int_0^{w_j}(MRI_j(x_j)-r)dx_j \tag{11}$$
$$\text{s.t.:} \quad \textstyle\sum_{j\in M_0} w_j \;\leq\; \sum_{i\in S}\omega^i,$$
$$w_j \;\geq\; 0, \qquad \text{for all } j\in M_0,$$

where

$$MRI_j(x_j) \;=\; \frac{\dfrac{\int_0^\infty te^{-x_j/\pi_j\cdot t/\alpha(S)}dF_{R_j}(t)}{\int_0^\infty e^{-x_j/\pi_j\cdot t/\alpha(S)}dF_{R_j}(t)} - \pi_j}{\pi_j}. \tag{12}$$

$MRI_j(x_j)$ represents the marginal return on investment in terms of certainty equivalents that coalition S receives from investing an additional dollar in the risky asset $j\in M_0$, when they have already invested x_j dollars in asset j. Note that the marginal return is decreasing[5] in the capital invested. Similarly, r represents the cost of capital, that is the marginal return of investing an additional dollar in the risk-free asset. In order to maximize the the total return, a coalition should invest in the asset with the largest marginal return on investment that exceeds the cost of capital. If such investments do not exist, the remaining capital should be invested in the risk-free asset. Hence, we can construct the optimal portfolio in the following way.

When no investments have been made, that is $x_j = 0$ for all $j\in M_0$, the marginal return of asset $j\in M_0$ equals

$$MRI_j(0) \;=\; \frac{E(R_j)-\pi_j}{\pi_j}.$$

Assuming that $MRI_{j_1}(0) > MRI_{j_2}(0) > \ldots > MRI_{j_m}(0)$, it is optimal to invest in asset j_1. As the capital x_{j_1} invested in asset j_1 increases, the marginal return $MRI_{j_1}(x_{j_1})$ decreases, and either one of the following three things will happen: the capital runs out, the marginal return becomes equal to the cost of capital, or the marginal benefit becomes equal to $MRI_{j_2}(0)$. In the first case, the current investments make up the optimal portfolio, while in the second case, the remaining capital is invested in the risk-free asset. In the third case, it is optimal to start investing in both asset j_1 and asset j_2. The quantities x_{j_1} and x_{j_2}, however, must be chosen such that the marginal returns of both assets remain equal to each other, that is $MRI_{j_1}(x_{j_1}) = MRI_{j_2}(x_{j_2})$. Again, as the investments increase, the marginal returns decrease simultaneously, and either of the following three things will happen: the capital runs out, the marginal return of both assets becomes equal to the cost of capital, or the marginal return becomes equal to $MRI_{j_3}(0)$. The procedure continues in a similar way until either the capital runs out, or the marginal return of all assets in the portfolio equals the cost of capital.

Let us illustrate this procedure with an example.

Example 2 Consider two individuals with $\alpha_1 = 0.1$ and $\alpha_2 = 0.05$, who can invest their individual capital of \$ 2 and \$ 4 in three risky assets and a risk-free asset. The risk-free interest

[5] This follows indirectly from Lemma 2 with $c = \frac{1}{\pi_j\alpha(S)}$.

rate is 10%. The future value of the risky assets is exponentially distributed with expected return of \$ 4, \$ 6, and \$ 8.1, respectively, while the current asset prices are \$ 3, \$ 5, and \$ 8, respectively. The marginal return on investment for asset j equals

$$MRI_j(x_j) \;=\; \frac{\alpha(S)E(R_j)}{\alpha(S)\pi_j + x_jE(R_j)} - 1$$

for all $x_j \geq 0$. The optimal portfolio for coalition $S = \{1,2\}$ is now constructed as follows. When $x_j = 0$ for $j = 1,2,3$ we have that

$$\begin{aligned} MRI_1(0) &= \tfrac{E(R_1)}{\pi_1} - 1 &= 0.333, \\ MRI_2(0) &= \tfrac{E(R_2)}{\pi_2} - 1 &= 0.200, \\ MRI_3(0) &= \tfrac{E(R_3)}{\pi_3} - 1 &= 0.013. \end{aligned}$$

Since asset 1 has the largest marginal return, coalition S starts with investing in asset 1 and increases this investment until $MRI_1(x_1) = MRI_2(0)$, which happens at $x_1 = 2.500$. They continue by investing in asset 1 and asset 2, such that $MRI_1(x_1) = MRI_2(x_2)$. This holds true if

$$x_1 \;=\; \alpha(S)\left(\frac{\pi_2}{E(R_2)} - \frac{\pi_1}{E(R_1)}\right) + x_2.$$

They increase the investments x_1 and x_2 until the available capital of \$ 6 is used up. Then $x_1 = 4.250$ and $x_2 = 1.750$. Note that for the current portfolio, the marginal return on investment $MRI_1(4.250) = MRI_2(1.750) = 0.1215$ still exceeds the cost of capital $r = 0.100$. Thus, coalition S would like to invest more in asset j_1 and asset j_2, but is constrained by its capital budget. Then the optimal investment portfolio equals $w^* = (4.250, 1.750, 0)$, or $q^* = (1.417, 0.350, 0)$ if stated in quantities.

Proposition 3 Let $\Gamma \in FG(N)$ be a financial game and let $q^* \in \mathbb{R}_+^{M_0}$ be the optimal portfolio for coalition $S \subset N$. Then
(a) $q_j^* = 0$ if $\frac{E(R_j)-\pi_j}{\pi_j} \leq r$.
(b) $q_j^* > 0$ if there exists $k \in M_0$ such that $q_k^* > 0$ and $\frac{E(R_j)-\pi_j}{\pi_j} > \frac{E(R_k)-\pi_k}{\pi_k}$.

PROOF: Let $\Gamma \in FG(N)$ and $S \subset N$. From (11) it follows that coalition S does certainly not invest any capital in asset j if $MRI_j(x_j) \leq r$ for all $x_j \geq 0$. Since the marginal return is strictly decreasing in x_j, we obtain that $w_j^* = 0$ if $MRI_j(0) < r$. Hence, $q_j^* = \frac{w_j^*}{\pi_j} = 0$ if $MRI_j(0) < r$, which proves (1).

In order to prove (2), let $MRI_{j_1}(0) > MRI_{j_2}(0) > \ldots > MRI_{j_m}(0)$. Recall that in the construction of the optimal portfolio, coalition S starts with investing in asset j_1, continues with investing in asset j_1 and asset j_2, and so on. Thus, if $MRI_j(0) > MRI_k(0)$ and $w_k^* > 0$, then coalition S must also invest in asset j. Hence, $w_j^* > 0$ and, consequently, $q_j^* = \frac{w_j^*}{\pi_j} > 0$.$\square$

Summarizing, a coalition will not invest in assets with an expected return on investment lower than the cost of capital, that is $\frac{E(R_j)-\pi_j}{\pi_j} \leq r$. Furthermore, it is the asset's expected return on investment that determines whether or not a coalition should invest in it. The variability in the asset's return only plays a role in determining the quantity that is invested. Consider, for example, the following two assets. Asset 1 has a price of \$ 90 and yields \$ 1000 with probability 0.55 and \$ -1000 with probability 0.45. Asset 2 has a price of \$ 90 and yields \$ 99 with certainty. Since the expected return of asset 1 exceeds the expected return of asset 2 by 1.11%, it is optimal to start investing in asset 1, although it involves (much) more risk than asset 2.

4.2 The Proportional Rule

The proportional rule is most commonly used by investment funds to allocate the returns to the participants in the fund. This means that if an investment fund generates a rate of return of 12%, each individual participant earns 12% on the amount of capital that he contributed to the fund. So, the individual rate of return does not depend on the amount of capital brought in. Izquierdo and Rafels (1996) and Borm et al. (1999) show that the proportional rule results in a core-allocation for deposit games, in which the rate of return is risk-free and dependent on the term and the amount of the bank deposit. For the class of financial games introduced in this section, however, the proportional allocation-rule does not perform as well.

Let $\Gamma \in FG(N)$ be a financial game and let $q^N \in \mathbb{R}^M_+$ denote the optimal portfolio. We start with considering the proportional rule for the corresponding TU-game (N, v_Γ). In terms of certainty equivalents, the rate of return of the optimal portfolio q^N equals

$$\frac{v_\Gamma(N) - \omega(N)}{\omega(N)},$$

where $\omega(N) = \sum_{k \in N} \omega_k$. The proportional rule ρ is then defined by

$$\rho_i \;=\; \omega_i \left(1 + \frac{v_\Gamma(N) - \omega(N)}{\omega(N)}\right) \;=\; \frac{\omega_i}{\omega(N)} v_\Gamma(N), \tag{13}$$

for all $i \in N$.

The proportional rule ρ, however, is not proportional in the sense that each individual $i \in N$ receives the random payoff $\frac{\omega_i}{\omega(N)} \sum_{j \in N} R_j q_j^N$. Since we considered the TU-game (N, v_Γ), the total return $\sum_{j \in M} R_j q_j^N$ is allocated Pareto optimally instead of proportionally, that is

$$r_i = \frac{\frac{1}{\alpha_i}}{\sum_{k \in N} \frac{1}{\alpha_k}}$$

instead of $r_i = \frac{\omega_i}{\omega(N)}$, for all $i \in N$. It is the certainty equivalent of this Pareto optimal allocation that is distributed proportionally by ρ. Note, however, that

$$r_i = \frac{\frac{1}{\alpha_i}}{\sum_{k \in N} \frac{1}{\alpha_k}} \;=\; \frac{\omega_i}{\omega(N)}$$

if and only if $\frac{\omega_i}{\alpha(\{i\})} = \frac{\omega_k}{\alpha(\{k\})}$ for all $i, k \in N$.

Theorem 4 Let $\Gamma \in FG(N)$. Then $\rho \in C(v_\Gamma)$ if and only if $\frac{\omega_i}{\alpha(\{i\})} = \frac{\omega_k}{\alpha(\{k\})}$ for all $i, k \in N$.

PROOF: Take $\Gamma \in FG(N)$. For all $S \subset N\mathbb{R}^M_+$ denote the optimal portfolio and let $\omega(S) = \sum_{i \in S} \omega_i$. If $\frac{\omega_i}{\alpha(\{i\})} = \frac{\omega_k}{\alpha(\{k\})}$ for all $i, k \in N$, then for all $S, T \subset N$ it holds true that

$$\frac{\omega(S)}{\alpha(S)} \;=\; \sum_{i \in S} \frac{\alpha(\{i\})}{\alpha(S)} \frac{\omega_i}{\alpha(\{i\})} \;=\; \sum_{k \in S} \frac{\alpha(\{k\})}{\alpha(T)} \frac{\omega_k}{\alpha(\{k\})} \;=\; \frac{\omega(T)}{\alpha(T)}.$$

For each $S \subset N$ we have that

$$\begin{aligned}
\sum_{i \in S} \rho_i &= \frac{\omega(S)}{\omega(N)} \alpha(N) \sum_{j \in M} \log \left(\int_0^\infty e^{-q_j^N t/\alpha(N)} dF_{R_j}(t) \right)^{-1} \\
&\geq \frac{\omega(S)}{\omega(N)} \alpha(N) \sum_{j \in M} \log \left(\int_0^\infty e^{-\frac{\omega(N)}{\omega(S)} q_j^S t/\alpha(N)} dF_{R_j}(t) \right)^{-1} \\
&= \alpha(S) \sum_{j \in M} \log \left(\int_0^\infty e^{-q_j^S t/\alpha(S)} dF_{R_j}(t) \right)^{-1} \\
&= v_\Gamma(S),
\end{aligned}$$

where the inequality follows from the fact that $\left(\frac{\omega(N)}{\omega(S)} q_j^S \right)_{j \in M}$ is a feasible portfolio for coalition N, and the second equality follows from $\frac{\omega(S)}{\omega(N)} \alpha(N) = \alpha(S)$. Hence, $\rho \in C(v_\Gamma)$.

Next, suppose that $\frac{\omega_{k_1}}{\alpha(\{k_1\})} < \frac{\omega_{k_2}}{\alpha(\{k_2\})}$ for some $k_1, k_2 \in N$. Since

$$\frac{\omega_{k_1}}{\alpha(\{k_2\})} < \frac{\omega(N)}{\alpha(N)} < \frac{\omega_{k_2}}{\alpha(\{k_2\})}$$

there exists $S \subset N$ such that $\frac{\omega(S)}{\alpha(S)} < \frac{\omega(N)}{\alpha(N)}$. Then

$$\begin{aligned}
\sum_{i \in S} \rho_i &= \alpha(S) \sum_{i \in S} \frac{\rho_i}{\alpha(S)} \\
&= \alpha(S) \frac{\omega(S)\alpha(N)}{\omega(N)\alpha(S)} \sum_{j \in M} \log \left(\int_0^\infty e^{-q_j^N t/\alpha(N)} dF_{R_j}(t) \right)^{-1} \\
&< \alpha(S) \sum_{j \in M} \log \left(\int_0^\infty e^{-\frac{\omega(S)\alpha(N)}{\omega(N)\alpha(S)} q_j^N t/\alpha(N)} dF_{R_j}(t) \right)^{-1} \\
&= \alpha(S) \sum_{j \in M} \log \left(\int_0^\infty e^{-\frac{\omega(S)}{\omega(N)} q_j^N t/\alpha(S)} dF_{R_j}(t) \right)^{-1} \\
&\leq \alpha(S) \sum_{j \in M} \log \left(\int_0^\infty e^{-q_j^S t/\alpha(S)} dF_{R_j}(t) \right)^{-1} \\
&= v_\Gamma(S),
\end{aligned}$$

where the first inequality follows from Jensen's inequality, $\frac{\omega(S)\alpha(N)}{\omega(N)\alpha(S)} < 1$, and R_j nondegenerate for at least one $j \in M$. The second inequality follows from the fact that $\left(\frac{\omega(S)}{\omega(N)} q_j^S\right)_{j \in M}$ is a feasible portfolio for coalition S. Hence, $\rho \notin C(v_\Gamma)$. □

5 Remarks

First, note that the results presented in this paper still go through if we replace mutual independency of $(P_j)_{j \in M}$ by the assumption that the covariance matrix of the retail prices $(P_j)_{j \in M}$ is negative definite.

Second, in this paper we only discussed price uncertainty, but the stochastic linear production game also applies to situations where there is uncertainty in the production process due to production losses. Production losses may occur when produced output does not satisfy prespecified (quality) standards. We can express these losses as a percentage of the production plan. Consider, for instance, a producer of audio and video tapes, and suppose that production losses are 2% for audio tapes and 0.7% for video tapes. Then given a feasible production plan $c \in \mathbb{R}_+^2$, the proceeds are $0.98 p_a c_a + 0.993 p_v c_v$.

Fall out of production, however, is generally uncertain at the start of production. Let $1 - X_j^i$ denote the stochastic percentage of production losses of commodity j when using technology i, and let p_j denote the deterministic retail price of good $j \in M$. Given a feasible production plan $c \in C(S)$, coalition $S \subset N$ obtains the payoff $\sum_{j \in M} \sum_{i \in S} p_j X_j^i c_j^i$. The corresponding stochastic cooperative game $(N, \{\mathcal{X}_S\}_{S \subset N}, (\succsim_i)_{i \in N})$ is given by

$$\mathcal{X}_S = \left\{ \sum_{j \in M} \sum_{i \in S} p_j X_j^i c_j^i \,\middle|\, \sum_{i \in S} c^i \in C(S) \right\}. \tag{14}$$

for all $S \subset N$, and $\succsim_i$ induced by U_i for each $i \in N$. Note that this model can easily be written in terms of a stochastic linear production game by considering c_j^i and c_j^k, $i \neq k$, as two different commodities. Hence, they are totally balanced as well.

Appendix

Lemma 1 Let $c \in \mathbb{R} \backslash \{0\}$ and let F be a probability distribution function corresponding to a non-degenerate random variable. Then the function h_c defined by $h_c(x) = \log \left(\int_0^\infty e^{xct} dF(t)\right)^{-1}$ for $x \geq 0$, is strictly concave in x.

PROOF: Since

$$\frac{dh_c}{dx} = -c \frac{\int_0^\infty t e^{xct} dF(t)}{\int_0^\infty e^{xct} dF(t)}$$

we obtain that

$$\begin{aligned}\frac{dh_c^2}{dx^2} &= -c^2\frac{\int_0^\infty t^2 e^{xct}dF(t)\int_0^\infty e^{xct}dF(t) - \left(\int_0^\infty te^{xct}dF(t)\right)^2}{\left(\int_0^\infty e^{xct}dF(t)\right)^2} \\ &= -c^2\int_0^\infty \left(t - \frac{\int_0^\infty \tau e^{xc\tau}dF(\tau)}{\int_0^\infty e^{xc\tau}dF(\tau)}\right)^2 \frac{e^{xct}}{\int_0^\infty e^{xc\tau}dF(\tau)}dF(t) \\ &\leq 0.\end{aligned}$$

Note that $\frac{dh_c^2}{dx^2}$ equals $-c$ times the variance of a random variable with density function

$$g(t) = \frac{e^{xct}}{\int_0^\infty e^{xc\tau}dF(\tau)}f(t),$$

with $f(t) = \frac{dF(t)}{dt}$. Since $\frac{dh_c^2}{dx^2} \leq 0$, it follows that h_c is concave. The lemma then follows from the observation that the inequality is binding if and only if F corresponds to a degenerate random variable. □

Lemma 2 Let $c \in \mathbb{R}\backslash\{0\}$ and let F be a probability distribution function corresponding to a non-degenerate random variable. Then the function h_c defined by

$$h_c(x) = \frac{\int_0^\infty te^{-xct}dF(t)}{\int_0^\infty e^{-xct}dF(t)}$$

for $x \geq 0$, is strictly decreasing in x.

PROOF: Since

$$\begin{aligned}\frac{dh_c}{dx} &= -c\frac{\int_0^\infty t^2 e^{-xct}dF(t)\int_0^\infty e^{-xct}dF(t) - \left(\int_0^\infty te^{-xct}dF(t)\right)^2}{\left(\int_0^\infty e^{-xct}dF(t)\right)^2} \\ &= -c\int_0^\infty \left(t - \frac{\int_0^\infty \tau e^{-xc\tau}dF(\tau)}{\int_0^\infty e^{-xc\tau}dF(\tau)}\right)^2 \frac{e^{-xct}}{\int_0^\infty e^{-xc\tau}dF(\tau)}dF(t) \\ &\leq 0.\end{aligned}$$

h_c is decreasing. Similar to the proof of Lemma 1, the result follows from the observation that the inequality is binding if and only if F corresponds to a degenerate random variable. □

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